an encounter is given by (4.5).

Let inequalities (3.1) and (3.6) hold, i.e.,

$$\delta+\beta\ln\left((q-\tau)\frac{\gamma}{l}\right)\leqslant\nu<\delta+(q-\tau)\gamma-l$$

We write the equation for the number s of (3.7):

$$\nu=\delta+(s-\tau)\gamma+\beta\ln((q-s)\gamma/l)$$

Then, we find from (3.9) that the law of mass variation has the form (4.5) with $\tau\leqslant t\leqslant s$, and with $s\leqslant t\leqslant p$,

$$m(t)=m(s)((q-t)/(q-s))^k,\quad k=\beta/|w|$$

Knowing the conditions for an encounter with any $a\geqslant 0$, we can find /4/ the value of the game, when the pay-off is the distance $|z(q)|$. In our example, the set $T_t^q(X)$ is not a stable bridge. This implies that termination of the game after the first instant of absorption /8/ is not possible for all initial positions, while the value of the game is not the same as the programmed max-min.

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Translated by D.E.B.

PMM U.S.S.R., Vol.51, No.2, pp.144-149, 1987
Printed in Great Britain

0021-8928/87 $10.00+0.00

CONSTRUCTION OF MIXED STRATEGIES ON THE BASIS OF STOCHASTIC PROGRAMS*

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The optimal control problem in the class of mixed strategies is considered, under the condition that the guaranteed result is minimized. An efficient method of constructing the optimal strategy by means of stochastic program synthesis is given. The results extend the theory given in /1-7/.

1. Formulation of the problem. We consider the object described by the differential equation

$$x^{\cdot}=A(t)x+f(t,u,v),\quad t_0\leqslant t\leqslant\vartheta,\quad u\in R,\quad v\in W \tag{1.1}$$

where x is the n-dimensional phase vector, u the r-dimensional control vector, v the s-dimensional noise vector, R and W are compacta, the matrix function $A(t)$ and vector function

**Prikl.Matem.Mekhan.*, 51, 2, 186-192, 1987

$f(t, u, v)$ are continuous, and t_0 and ϑ are fixed.

We are given the performance factor $\gamma = |x[\vartheta]|$, where $|x|$ is the Euclidean norm of x.

Along with the object x of (1.1) we consider the model y, connected to the control organ. Its phase state is defined by the n-dimensional vector $y[t]$.

We define a mixed strategy as a combination

$$S^u = \{R(\cdot),\ p(\cdot);\ R^*(\cdot),\ p^*(\cdot);\ W^*(\cdot),\ q^*(\cdot)\} \tag{1.2}$$

of sets and functions

$$R(\cdot) = \{R(\varepsilon) = \{u^{[l]} \in R,\ \ l = 1, \ldots, N_\varepsilon\},\ \ \varepsilon > 0\},$$
$$R^*(\cdot) = R(\cdot)$$
$$p(\cdot) = \Big\{p_l(t, x, y, \varepsilon) \geqslant 0,\ \ \sum_{l=1}^{N_\varepsilon} p_l(t, x, y, \varepsilon) = 1\Big\}$$
$$p^*(\cdot) = \Big\{p_l^*(t, x, y, \varepsilon) \geqslant 0,\ \ \sum_{l=1}^{N_\varepsilon} p_l^* = 1\Big\}$$
$$W^*(\cdot) = \{W^*(\varepsilon) = \{v^{[m]} \in W,\ m = 1, \ldots, M_\varepsilon\}\}$$
$$q^*(\cdot) = \Big\{q_m^*(t, x, y, \varepsilon) \geqslant 0,\ \ \sum_{m=1}^{M_\varepsilon} q_m^* = 1\Big\}$$

The control law U, corresponding in the interval $[t_*, \vartheta]$ to strategy S^u of (1.2), is defined as the combination

$$U = \{S^u;\ \varepsilon > 0;\ \Delta\{t_i\}\},\quad t_* \in [t_0, \vartheta) \tag{1.3}$$
$$\Delta\{t_i\} = \{t_1 = t_*, \ldots, t_i < t_{i+1},\ t_{i+1} = \vartheta\} \tag{1.4}$$

The basic probability space $\{\Omega, F, P\}$, is constructed in the standard way on the basis of the functions $p(\cdot)$, $p^*(\cdot)$, $q^*(\cdot)$ and the properties of the random noise

$$v[t_*[\cdot]\vartheta, \cdot) = \{v[t, \omega] \in W,\quad t_* \leqslant t < \vartheta,\ \omega \in \Omega\} \tag{1.5}$$

The control law U of (1.3) jointly with the noise $v[\cdot]$ of (1.5) generates from the initial positions $\{t_*, x_*\}$ and $\{t_*, y_*\}$ random motions of the object x and of model y, which are found as solutions of the stepped equations

$$x^\cdot[t, \omega] = A(t)\,x[t, \omega] + f(t,\ u[t_i, \omega],\ v[t, \omega]),\ t_i \leqslant t < t_{i+1}, \tag{1.6}$$
$$i = 1, \ldots, k$$

$$y^\cdot[t, \omega] = A(t)\,y[t, \omega] + \sum_{l,\,m=1}^{N_\varepsilon, M_\varepsilon} f(t, u^{[l]}, v^{[m]})\,p_l^*(t_i, x[t_i, \omega], \tag{1.7}$$
$$y[t_i, \omega], \varepsilon)\,q_m^*(t_i, x[t_i, \omega], y[t_i, \omega], \varepsilon),\ \ t_i \leqslant t < t_{i+1}$$

When forming the motion $x[t, \omega]$ in accordance with (1.6), a random test is made at each step at the instant t_i on the choice of the vector $u[t_i, \omega] \in R(\varepsilon)$ with conditional probability

$$P(u[t_i, \omega] = u^{[l]} \mid x[t_i, \omega],\ y[t_i, \omega]) = p_l(t_i,\ x[t_i\,\omega],\ y[t_i, \omega],\ \varepsilon)$$

We assume that the noise is stochastically independent of the control at each step, i.e.,

$$P(v[t, \omega] \in C \mid x[t_i, \omega],\ y[t_i, \omega],\ u[t_i, \omega]) =$$
$$P(v[t, \omega] \in C \mid x[t_i, \omega],\ y[t_i, \omega])$$

We define the guaranteed result for U, $\{t_*, x_*\}$, $\{t_*, y_*\}$ and $\beta \in [0, 1)$ as the quantity

$$\rho(U;\ t_*,\ x_*,\ y_*;\ \beta) = \min \alpha \tag{1.8}$$

where the values α satisfy the condition $P(\gamma = |x[\vartheta, \omega]| \leqslant \alpha) \geqslant \beta$ for any admissible noise (1.5).

For strategy S^u, we define the guaranteed result for $\{t_*, x_*\}$ as the number

$$\rho(S^u; t_*, x_*) = \lim_{\beta \to 1} \overline{\lim_{\varepsilon \to 0}} \lim_{\zeta \to 0} \sup_{|y_* - x_*| \leqslant \zeta} \lim_{\delta \to 0} \sup_{\Delta_\delta} \rho(U; t_*, x_*, y_*; \beta)$$

($\Delta_\delta = \Delta\{t_i\}$ is the division (1.4) with step $\max_i |t_{i+1} - t_i| \leqslant \delta$).

We shall call the strategy S_0^u optimal if we have

$$\rho(S_0^u; t_*, x_*) = \min_{S^u} \rho(S^u; t_*, x_*) = \rho^\circ(t_*, x_*) \tag{1.9}$$

for all positions $\{t_*, x_*\} \in G$, where G is a pre-assigned bounded closed domain which satisfies the condition: given any initial position $\{t_*, x_*\} \in G$ $\{t_*, y_*\} \in G$, we have for every possible motion $x[t_*[\cdot]\vartheta, \omega]$, $y[t_*[\cdot]\vartheta, \omega]$ the inclusion $\{t, x[t, \omega]\} \in G$, $\{t, y[t, \omega]\} \in G$ for all $t \in [t_*, \vartheta]$, $\omega \in \Omega$ (/5/, p.67). The quantity $\rho^\circ(t_*, x_*)$ of (1.9) is the optimal guaranteed result.

Our problem is to evaluate $\rho^\circ(t_*, x_*)$ and to construct a strategy S_0^u.

We have:

Theorem 1.1. An optimal universal strategy $S_0^u = \{S_0^u(t, x, y, \varepsilon)\}$ exists.

This means that, for any $\eta > 0$ and $\beta \in [0, 1)$, we can indicate $\varepsilon(\eta, \beta) > 0$, $\zeta(\eta, \beta, \varepsilon) > 0$, $\delta(\eta, \beta, \varepsilon, \zeta) > 0$ in such a way that, for the motion $x[t_*[\cdot]\vartheta, \cdot]$, generated from position $\{t_*, x_*\} \in G$ by control law $U^\circ \doteq \{S_0^v; \varepsilon, \Delta_\delta\}$, we have the inequality

$$P(|x[\vartheta, \omega]| \leqslant \rho^\circ(t_*, x_*) + \eta) \geqslant \beta$$

if $\varepsilon \leqslant \varepsilon(\eta, \beta)$, $|y_* - x_*| \leqslant \zeta(\eta, \beta, \varepsilon)$, $\delta \leqslant \delta(\eta, \beta, \varepsilon, \zeta)$, no matter what the noise (1.5). The quantity ρ° is the least number that satisfies this condition. Below we describe the approximation of the function $\rho^\circ(t, x)$ based on the method of stochastic programmed **synthesis** /5/, and on this basis we construct the strategy S_0^u.

2. Closeness of motions of the object x and model y. Let the set $R_\eta = \{u^{[l]} \in R, l = 1, \ldots, N\}$ be such that, for any $u \in R$, there exists $u^{[l]} \in R_\eta$, $|u - u^{[l]}| \leqslant \eta$. Similarly, $W_\eta = \{v^{[m]} \in W, m = 1, \ldots, M\}$ is the set such that, for any $v \in W$, there exists $v^{[m]} \in W_\eta$, $|v - v^{[m]}| \leqslant \eta$. We form the n-dimensional vector $r = x - y$. Let the positions $\{\tau_*, x[\tau_*]\} \in G$ and $\{\tau_*, y[\tau_*]\} \in G$ be found. We construct the motion $y[t]$ in accordance with the equation

$$y^\cdot[t] = A(t)y[t] + \sum_{l, m=1}^{N, M} f(t, u^{[l]}, v^{[m]})\, p_l^* q_m^*, \quad \tau_* \leqslant t < \tau^* \tag{2.1}$$

$$u^{[l]} \in R_\eta,\ v^{[m]} \in W_\eta,\ p_l^* \geqslant 0,\ q_m^* \geqslant 0$$

$$\sum_{l=1}^{N} p_l^* = 1, \quad \sum_{m=1}^{M} q_m^* = 1, \quad \tau^* \in (\tau_*, \vartheta]$$

We choose the sets of numbers $\{p_l^\circ \geqslant 0,\ l = 1, \ldots, N,$

$$\sum_l p_l^\circ = 1\} \text{ and } \{q_m^{*\circ} \geqslant 0,\ \sum_m q_m^{*\circ} = 1\},$$

which satisfy the equations

$$\max_q \sum_{l, m=1}^{N, M} \langle r[\tau_*] \cdot f(\tau_*, u^{[l]}, v^{[m]})\, p_l^\circ q_m \rangle = \min_p \operatorname{Idem}(p^\circ \to p) \tag{2.2}$$

$$\min_{p^*} \sum_{l, m=1}^{N, M} \langle r[\tau_*] \cdot f(\tau_*, u^{[l]}, v^{[m]})\, p_l^* q_m^{*\circ} \rangle = \max_{q^*} \operatorname{Idem}(q^{*\circ} \to q^*) \tag{2.3}$$

where $r[\tau_*] = x[\tau_*] - y[\tau_*]$.

Here and below, **Idem** on the right-hand side of an equation signifies an expression which is the same as on the left-hand side with the change of symbols indicated in the parentheses; $\langle r \cdot f \rangle$ is the scalar product.

The set of numbers $\{p_l^\circ\}, \{q_m^{*\circ}\}$ which satisfy conditions (2.2), (2.3) with fixed $\{u^{[l]}\}$, $\{v^{[m]}\}$, and $r[\tau_*] = r_*$, may not be unique. Given $\{u^{[l]}\}$, $\{v^{[m]}\}$, and r_*, we choose a set $\{p_l^\circ\}$ and a set $\{q_m^{*\circ}\}$. Let the probability $P(u = u^{[l]} \in R_\eta)$ be equal to the number p_l° of the set $\{p_1^\circ, \ldots, p_N^\circ\}$, which satisfies condition (2.2).

We introduce the function

$$\nu(t, x, y) = |x - y|^2 \exp\{-2\lambda(t - t_0)\};$$

$$\lambda = \max_{t_0 \leqslant t \leqslant \vartheta} |A(t)|, \quad |A(t)| = \max_{|x| \leqslant 1} |A(t)x|$$

Lemma 2.1. Given any $\varepsilon > 0$, there exist $\delta(\varepsilon) > 0$ and $\eta(\varepsilon) > 0$ such that we have the following claim.

Let $\tau^* \in (\tau_*, \vartheta]$ satisfy the condition $\tau^* - \tau_* \leqslant \delta(\varepsilon)$ and $\eta \leqslant \eta(\varepsilon)$. The motion $y[\tau_*[\cdot]\tau^*] = \{y[t], \tau_* \leqslant t \leqslant \tau^*\}$ starts from the initial position $\{\tau_*, y[\tau_*]\} \in G$ by the set $\{W_\eta, q_m^{*\circ}\}$ in the pair with any set $\{u^{[l]}, p_l^*\}$. Let $x[\tau_*[\cdot]\tau^*, \cdot] = \{x[t, \omega^*], \tau_* \leqslant t \leqslant \tau^*, \omega^* \in \Omega^*\}$ be the random motion generated according to a scheme similar to (1.6) with $t_i = \tau_*$, $t_{i+1} = \tau^*$, $x[t_1, \omega^*] = x[\tau_*]$ for the chosen set $\{R_\eta, p_l^\circ\}$ and any admissible noise $v[t, \omega^*]$. Then we have the inequality

$$M\{\nu(t, x[t, \omega^*], y[t])\} \leqslant \nu(\tau_*, x[\tau_*], y[\tau_*]) + \varepsilon \cdot (t - \tau_*)$$

for all $t \in [\tau_*, \tau^*]$. Here, $\{\Omega^*, F^*, P^*\}$ is the appropriate auxiliary probability space, and

$M\{\ldots\}$ is the expectation.

3. Programmed extremum. We take the model w, described by the differential equation

$$w^{\cdot} = A(t)w + \sum_{l,m=1}^{N_*, M_*} f(t, u_*^{[l]}, v_*^{[m]}) p_{*l} q_{*m} \tag{3.1}$$
$$u_*^{[l]} \in R_*, \quad v_*^{[m]} \in W_*, \quad p_{*l} \geqslant 0, \quad q_{*m} \geqslant 0$$
$$\sum_l p_{*l} = 1, \quad \sum_m q_{*m} = 1, \quad \tau_* \leqslant t \leqslant \vartheta$$

In (3.1), the values N_*, M_* and sets R_*, W_* are fixed. Let k be a given positive integer. We designate the division $\Delta\{\tau_j\} = \{\tau_1 = \tau_*, \ldots \tau_j < \tau_{j+1}, \quad j = 1, \ldots, k, \quad \tau_{k+1} = \vartheta\}$. With it we connect the random quantities $\{\xi_1, \ldots, \xi_k\}$, independent in aggregate, each of which is uniformly distributed in the half-interval $0 \leqslant \xi_j < 1$. Each set $\{\xi_1, \ldots, \xi_k\}$ is treated as an elementary event ω_* in probability space $\{\Omega_*, B_*, P_*\}$, where $\Omega_* = \{\omega_*\}$ is the unit cube in k-dimensional space, B_* is the Borel σ-algebra for this cube, $P_* = \{P_*(B)\}$ is the Lebesgue measure, $B \in B_*$. The programmed extremum (/5/, p.291) is given by the equation

$$e(\tau_*, w_*, \Delta\{\tau_j\}) = \sup_{\|l(\cdot)\| \leqslant 1} \Big[\langle m_* \cdot X(\vartheta, \tau_*) w_* \rangle + \tag{3.2}$$
$$M\Big\{ \int_{\tau_*}^{\vartheta} \min_{p_*} \max_{q_*} \Big\langle m(\tau, \omega_*) \cdot X(\vartheta, \tau) \times$$
$$\sum_{l,m=1}^{N_*, M_*} f(\tau, u_*^{[l]}, v_*^{[m]}) p_{*l} q_{*m} \Big\rangle d\tau \Big\}\Big]$$

Here, $l(\cdot) = \{l(\omega_*), \ \omega_* \in \Omega_*\}$ is an n-dimensional vector random quantity, given in $\{\Omega_*, B_*, P_*\}$; $X(t, \tau)$ is the fundamental matrix of solutions of the equation $w^{\cdot} = A(t)w$. Moreover, in (3.2),

$$\| l(\cdot) \| = (M\{| l(\omega_*)|^2\})^{1/2}, \quad m_* = M\{l(\omega_*)\}$$
$$m(\tau, \omega_*) = M\{l[\xi_1, \ldots, \xi_k] \mid \xi_1, \ldots, \xi_j\},$$
$$\tau \in [\tau_j, \tau_{j+1})$$

We introduce the quantity

$$\varkappa(\tau_*, m_*) = \sup_{\Delta\{\tau_j\}} \varkappa(\tau_*, m_*, \Delta\{\tau_j\}) = \sup_{\Delta\{\tau_j\}} \sup_{\|b(\cdot)\| \leqslant (1-|m_*|^2)^{1/2}} M \times \tag{3.3}$$
$$\Big\{ \int_{\tau_*}^{\vartheta} \min_{p_*} \max_{q_*} \Big\langle (m_* + n(\tau, \omega_*)) \times$$
$$X\Big(\vartheta, \tau \sum_{l,m=1}^{N_*, M_*} f(\tau, u_*^{[l]}, v_*^{[m]}) \cdot p_{*l} q_{*m} \Big\rangle d\tau \Big\}$$
$$n(\tau, \omega_*) = M\{b[\xi_1, \ldots, \xi_k] \mid \xi_1, \ldots, \xi_j\}, \quad \tau_j \leqslant \tau < \tau_{j+1}.$$

On the basis of (3.2), (3.3), we construct the function (the prime denotes transposition)

$$\rho^{\varepsilon}(\tau_*, y_*) = -\beta(\varepsilon, \tau_*)(1 + | X'(\vartheta, \tau_*) m_*^{\circ}(\tau_*, y_*, \varepsilon)|^2)^{1/2} + \tag{3.4}$$
$$\langle m_*^{\circ}(\tau_*, y_*, \varepsilon)' X(\vartheta, \tau_*) y_* \rangle + \varkappa(\tau_*, m_*^{\circ}(\tau_*, y_*, \varepsilon)) =$$
$$\max_{|m_*| \leqslant 1} \operatorname{Idem}(m_*^{\circ}(\tau_*, y_*, \varepsilon) \to m_*)$$

$$\beta^2(\varepsilon, \tau_*) = \varepsilon + \varepsilon \exp\{2\lambda(\tau_* - t_0)\}$$

Noting that the function $\varkappa(\tau_*, m_*)$ of (3.3) is concave with respect to m_*, we can show that $m_*^{\circ}(\tau_*, y_*, \varepsilon)$ of (3.4) is a unit vector. Hence the vector function $m_*^{\circ}(\tau_*, y_*, \varepsilon)$ is continuous with respect to τ_* and y_*, while the function $\rho^{\varepsilon}(\tau_*, y_*)$ of (3.4) is differentiable with respect to y_*. Hence there exists the vector gradient

$$\partial\rho^{\varepsilon}(\tau_*, y_*)/\partial y_* = X'(\vartheta, \tau_*) m_*^{\circ}(\tau_*, y_*, \varepsilon) \tag{3.5}$$

which is continuous with respect to τ_* and y_*.

4. Guaranteed result. We choose the set of quantities

$$\{p_l^{*\varepsilon\circ}, \ l = 1, \ldots, R^{\varepsilon}\}$$

which satisfies for given τ_*, y_*, and $\varepsilon > 0$, the condition

$$\Big\langle \frac{\partial\rho^{\varepsilon}(\tau_*, y_*)}{\partial y_*} \sum_{l,m=1}^{N_\varepsilon, M_\varepsilon} f(\tau_*, u_*^{[l]}, v_*^{[m]}) p_l^{*\varepsilon\circ} q_m^{*\circ} \Big\rangle = \min_{p^*} \max_{q^*} \operatorname{Idem}(p^{*\varepsilon\circ} \to p^*, \ q^{*\circ} \to q^*) \tag{4.1}$$

$$\text{for } p_l^* \geqslant 0,\ q_m^* \geqslant 0,\ \sum_l p_l^* = 1,\ \sum_m q_m^* = 1.$$

Here and below, we assume that $u^{[l]} \in R^\varepsilon$, $v^{[m]} \in W^\varepsilon$, where $R^\varepsilon = R_\eta$, $W^\varepsilon = W_\eta$ satisfy the conditions of Lemma 2.1 for the given $\varepsilon > 0$.

We can prove the following property of u-stability.

Lemma 4.1. Given any $\alpha > 0$, we can indicate $\varepsilon(\alpha) > 0$ and $\delta(\alpha, \varepsilon) > 0$ such that, for any $\{\tau_*, y_*\} \in G$ and $\tau^* > \tau_*$, $|\tau^* - \tau_*| \leqslant \delta(\alpha, \varepsilon)$, the set $\{R^\varepsilon, p_l^{*\varepsilon^\circ}\}$ in combination with any set $\{W^\varepsilon, q_m^*\}$, generates from the position $\{\tau_*, y_*\}$ the motion $y[\tau_*[\cdot]\tau^*] = \{y[t], \tau_* \leqslant t \leqslant \tau^*\}$, for which we have the inequality

$$\rho^\varepsilon(\tau^*, y[\tau^*]) \leqslant \rho^\varepsilon(\tau_*, y_*) + \alpha\cdot(\tau^* - \tau_*),$$

provided that $\varepsilon \leqslant \varepsilon(\alpha)$, $\delta \leqslant \delta(\alpha, \varepsilon)$.

We see from this that, given any $\zeta^* > 0$, we can find an $\varepsilon(\zeta^*) > 0$ and $\delta(\zeta^*, \varepsilon) > 0$ such that, if division (1.4) satisfies the condition $\max_i |t_{i+1} - t_i| \leqslant \delta(\zeta^*, \varepsilon)$, and the motion $y[t_*[\cdot]\vartheta, \omega]$ is constructed with steps $t_i \leqslant t \leqslant t_{i+1}$, in such a way that $\{p_l^{*\varepsilon^\circ}\}$ are chosen from the conditions of Lemma 4.1, and $\{q_m^*\}$ arbitrarily then we have the inequality

$$|y[\vartheta, \omega]| \leqslant \rho^\varepsilon(t_*, y_*) + \zeta^*$$

provided $\varepsilon \leqslant \varepsilon(\zeta^*)$.

We construct the strategy $S_e^u = \{R_e(\cdot),\ p_e(\cdot);\ R_e^*(\cdot),\ p_e^*(\cdot);\ \dot W_e^*(\cdot),\ q_e^*(\cdot)\}$ (1.4), which we shall call extremal. We use the motion $y[t_*[\cdot]\vartheta, \omega]$ of model y as a guide to the motion of object x. Let $R_e(\cdot)$, $p_e(\cdot)$ be the rule which associates with possible values t°, x°, y°, and $\varepsilon^\circ > 0$ the sets $\{u^{[l]}, p_l^\circ, l = 1, \ldots, N_\varepsilon\}$, satisfying the conditions of Lemma 2.1 for $\tau_* = t^\circ$, $x[\tau_*] = x^\circ$, $y[\tau_*] = y^\circ$, $\varepsilon = \varepsilon^\circ$. Further, let $W_e^*(\cdot)$, $q_e^*(\cdot)$ be the rule which designates the sets $\{v^{[m]}, q_m^{*\circ}, m = 1, \ldots, M_\varepsilon\}$, which satisfy the conditions of Lemma 2.1. Finally, $R_e^*(\cdot)$, $p_e^*(\cdot)$ is the rule which designates $\{u^{[l]}, p_e^{*\varepsilon^\circ}, l = 1, \ldots, N_\varepsilon\}$, satisfying condition (4.1). It can be shown that functions $p_e(\cdot), p_e^*(\cdot), q_e^*(\cdot)$ can be chosen to be measurable with respect to x, y.

For the motions $x[t_*[\cdot]\vartheta, \cdot]$ and $y[t_*[\cdot]\vartheta, \cdot]$, which are formed by strategy S_e^u in accordance with scheme (1.6), (1.7) at each step $t_i \leqslant t \leqslant t_{i+1}$, we have the inequality

$$\begin{gathered} M\{\nu(t_{i+1}, x[t_{i+1}, \omega], y[t_{i+1}, \omega]) \mid x[t_i, \omega], y[t_i, \omega]\} \leqslant \\ \nu(t_i, x[t_i, \omega], y[t_i, \omega]) + \varepsilon\cdot(t_{i+1} - t_i) \end{gathered} \tag{4.2}$$

where $M\{\cdots \mid \cdots\}$ is the conditional expectation. By the expression for repeated expectations (/9/, p.55) we find, by (4.2),

$$M\{\nu(t_{i+1}, x[t_{i+1}, \omega], y[t_{i+1}, \omega])\} \leqslant \mathrm{Idem}(t_{i+1} \to t_i) + \varepsilon\cdot(t_{i+1} - t_i) \tag{4.3}$$

From (4.3), using Chebyshev's inequality (/9/, p.51), we obtain the following assertion.

Given the initial position $\{t_*, x_*\} \in G$, and any $\eta > 0$ and $\beta \in [0, 1)$, we can find $\varepsilon(\eta, \beta) > 0$, $\zeta(\eta, \beta, \varepsilon) > 0$, and $\delta(\eta, \beta, \varepsilon, \zeta) > 0$ such that, for every motion $x[t_*[\cdot]\vartheta, \cdot]$, formed from position $\{t_*, x_*\}$ by the above method with $\varepsilon \leqslant \varepsilon(\eta, \beta)$, $|y_* - x_*| \leqslant \zeta(\eta, \beta, \varepsilon)$ and with division Δ_δ, satisfying the condition $\max_i |t_{i+1} - t_i| \leqslant \delta(\eta, \beta, \varepsilon, \zeta)$, we have the inequality

$$P(|x[\vartheta, \omega]| \leqslant \rho^\varepsilon(t_*, x_*) + \eta) > \beta \tag{4.4}$$

no matter what the noise (1.5).

Let $U_e = \{S_e^u; \varepsilon; \Delta_\delta\}$ be the law (1.3), corresponding to strategy S_e^u. Then, in accordance with (4.4), the guaranteed result $\rho(U_e)$ of (1.8) satisfies the inequality

$$\rho(U_e; t_*, x_*, y_*; \beta) \leqslant \rho^\varepsilon(t_*, x_*) + \eta^* \tag{4.5}$$

provided that $\varepsilon \leqslant \varepsilon(\eta^*, \beta), \delta \leqslant \delta(\eta^*, \beta, \varepsilon, \zeta)$, $|y_* - x_*| \leqslant \zeta(\eta^*, \beta, \varepsilon)$.

5. Optimal guaranteed results. We take the model z, described by an equation similar to (2.1) with y replaced by z. We can prove a Lemma similar to Lemma 2.1 concerning the closeness of the motions of the object x and model z, where the sets $\{W_\eta, q_m^\circ\}$ are chosen for the object, and sets $\{R_\eta, p_l^{*\circ}\}$ for the model. We have the property of v-stability of the function $\rho_\varepsilon(\tau, \{w_1, \ldots, w_n\}) = e(\tau, \{w_1, \ldots, w_n\}, \Delta\{\tau_j\})$.

Lemma 5.1. For any $\alpha > 0$ we can find $\varepsilon(\alpha) > 0$ and $\delta(\alpha, \varepsilon) > 0$ such that, for any $\{\tau_*, w_*\}, \tau^* > \tau_*$, $|\tau^* - \tau_*| \leqslant \delta(\alpha, \varepsilon)$ and the set $\{p_{*l}\}$ there is a set $\{q_{*m}^\circ\}$, which in combination with $\{p_{*l}\}$, generates from position $\{\tau_*, w_*\}$ a position $\{\tau^*, w[\tau^*] = w^*\}$ such that we have the inequality

$$\rho_\varepsilon(\tau^*, \{w_1^*, \ldots, w_{n+1}^*\}) \geqslant \rho_\varepsilon(\tau_*, \{w_{1*}, \ldots, w_{n+1*}\}) - \alpha\cdot(\tau^* - \tau_*)$$

if $\varepsilon \leqslant \varepsilon(\alpha)$, $\delta \leqslant \delta(\alpha, \varepsilon)$

Using the motion of model z as a guide to the motion of object x, we can find an appropriate law of formation of the noise V_e for the object so that our claim holds. For any motion

$x[t_*[\cdot]\vartheta,\cdot]$, generated from position $\{t_*, x_*\} \in G$ by any law U (in accordance with scheme (1.6)) and law V_ε with a suitable particular division (1.4), we shall have the estimate

$$P(|x[\vartheta,\omega]| \geqslant \rho_\varepsilon(t_*, x_*) - \eta_*) > \beta \tag{5.1}$$

for $\varepsilon \leqslant \varepsilon(\eta_*, \beta)$, $\delta \leqslant \delta(\eta_*, \beta, \varepsilon)$.

We have the inequality

$$|\rho^\varepsilon(t_*, x_*) - \rho_\varepsilon(t_*, x_*)| \leqslant \psi(\varepsilon) \tag{5.2}$$

($\lim \psi(\varepsilon) = 0$ as $\varepsilon \to 0$)

Using (4.5) and (5.2), we find that

$$\rho(U_e; t_*, x_*, y_*; \beta) \leqslant \rho_\varepsilon(t_*, x_*) + \chi$$

for $\varepsilon \leqslant \varepsilon(\chi, \beta)$, $\delta \leqslant \delta(\chi, \beta, \varepsilon)$.

At the same time, according to (5.1), there is no admissible law U (1.3) which can guarantee the inequality

$$\rho(U; t_*, x_*, y_*; \beta) < \rho_\varepsilon(t_*, x_*) - \alpha$$

for values of β close to unity, and $\alpha > 0$.

From these inequalities we find that

$$\rho^*(t_*, x_*) = \lim_{\varepsilon \to 0} \rho^\varepsilon(t_*, x_*) \tag{5.3}$$

and that we have the following.

Theorem 5.1. The quantity $\rho^*(t_*, x_*)$ of (5.3) is the optimal guaranteed result $\rho^\circ(t_*, x_*)$ of (1.9) for any position $\{t_*, x_*\} \in G$. Strategy S_ε^u is the optimal strategy S_ε^u of (1.9).

Notice in conclusion that, to evaluate $\varkappa(\tau_*, m_*, \Delta\{\tau_j\})$ of (3.3) we can use the following procedure, which follows at once from the definition of this quantity. We evaluate the function

$$\psi_k(m_*) = I(\tau_k, \vartheta) = \int_{\tau_k}^{\vartheta} \min_{p_*} \max_{q_*} < m_* X(\vartheta, \tau) \sum_{l,\, m=1}^{N_*,\, M_*} f(\tau, u_*^{[l]}, v_*^{[m]}) \times p_{*l} q_{*m} > d\tau$$

Let the function $\varphi_k(m_*)$ be the upper concave hull for $\psi_k(m_*)$ with $|m_*| \leqslant 1$. The further construction of the functions $\varphi_i(m_*)$ $(i = k-1, k-2, \ldots, 1)$ is made in steps. Let the function $\varphi_i(m_*)$ $(i = k, \ldots, 2)$ be constructed. We evaluate the function

$$\psi_{i-1}(m_*) = \varphi_i(m_*) + I(\tau_{i-1}, \tau_i)$$

The function $\varphi_{i-1}(m_*)$ is constructed as the upper concave hull for $\psi_{i-1}(m_*)$ for $|m_*| \leqslant 1$. The function $\varphi_1(m_*)$ is equal to $\varkappa(\tau_*, m_*, \Delta\{\tau_j\})$. This procedure amounts to a sequence of convex programming problems.

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Translated by D.E.B.